Lecture 07

1. Solution manifolds in Jet space and its local diffeomorphic transformation

Let M be an open subset of $\mathbb{R}^n$ or more generally N-manifold , $F = (F_1, \ldots, F_l)$ be a system of $\mathcal{C}^\infty$ functions for positive integers $l < N$ and $\mathcal{S}_F$ be a zero set of $F = \{x \in \mathbb{R}^N : F_1(x) = \cdots = F_l(x) = 0\}$. Assume F is of maximal rank on $\mathcal{S}_F$ i.e. $\left(\frac{\partial F_\nu}{\partial x_\mu}\right)$ is of rank l or equivalently $dF_1, \ldots, dF_l$ are linearly independent on $\mathcal{S}_F$. Then $\mathcal{S}_F$ is a $\mathcal{C}^\infty$ manifold.

Proposition 1.1. *A smooth function f is defined on M. f vanishes on $\mathcal{S}_F$ if and only if $f = Q_1F_1 + \cdots + Q_lF_l$ for some $\mathcal{C}^\infty$ functions $Q_1, \ldots, Q_l$, i.e. f belongs to the ideal generated by $F_1, \ldots, F_l$ in the ring of $\mathcal{C}^\infty$ functions.*

Let X be an open subset of $\mathbb{R}^p$ and $U := \{(u^1, \ldots, u^q)\}$ be an open set in $\mathbb{R}^q$. Suppose there is a function f such that $u := f(x)$ for $x \in X$ and $u \in U$. The graph $\Gamma_f = \{(x, f(x)) \in X \times U\}$ is a p-dimensional submanifold of $X \times U$.
Let g be a local diffeomorphism $X \times U \to X \times U$. We let

$$g(\Gamma_f) = \{(\tilde{x}, \tilde{u}) = g(x, u) : (x, u) \in \Gamma_f\} := \Gamma_{\tilde{f}}.$$

Note that we consider only the infinitesimal transform of the identity component so that the graph of the function $u = f(x)$ is transformed by a diffeomorphism g to define graph of another function $\tilde{u} = \tilde{f}(\tilde{x})$. We write

$$g \circ f := \tilde{f}$$

, which we call the *transform* of f by g.

Example 1.2. Let $p = q = 1$, $X = \mathbb{R}$ and $G = SO(2)$[1]. Take the rotation $\Theta \in G$ as our diffeomorphic transformation. Then $\Theta(x, u) = (x \cos\theta - u \sin\theta, x \sin\theta + u \cos\theta) = (\tilde{x}, \tilde{u})$. Consider the graphs $u = ax + b = f(x)$. Substituting $u = -\tilde{x}\sin\theta + \tilde{u}\cos\theta$ and $x = \tilde{x} + \tilde{u}\sin\theta$ for $u = ax + b$, we have the graph $\tilde{u} = \frac{a\cos\theta + \sin\theta}{\cos\theta - a\sin\theta}\tilde{x} + b := \tilde{f}(\tilde{x})$.

Definition 1.3. For $x \in (x^1, \ldots, x^p) \in X$ and $u \in (u^1, \ldots, u^q) \in U$, the n-th jet space of $X \times U$ is

$$X \times U^{(n)} := \{(x, u^{(n)})\}$$

, which is endowed with Euclidean structure and smooth topology.

Definition 1.4. Given a system of partial differential equations of order n

$$\Delta_\nu(x, u^{(n)} = 0, \qquad \nu = 1, 2, \ldots, l$$

, where $\Delta = (\Delta_1, \ldots, \Delta_l)$, the system of $\mathcal{C}^\infty$ functions defined on $X \times U^{(n)}$, We define

$$\mathcal{S}_\Delta := \text{ zero set of } \Delta \text{ i.e. } \{\Delta = 0\}.$$

Remark 1.5. We only consider the case for which $\mathcal{S}_\Delta$ is smooth manifold i.e. $d\Delta_1, \ldots, d\Delta_l$ is of maximal rank.

[1] $O(2)$ has two components with the signature of the determinant ± 1. $SO(2)$ is the identity component of the two.

Hence we have the following equivalent notions.

$$u = f(x) \text{ is a solution of } \Delta = 0 \tag{1.1}$$
$$\iff \quad \Delta_\nu(x, f^{(n)}(x)) = 0, \quad \nu = 1, 2, \ldots, l \tag{1.2}$$
$$\iff \quad (x, f^{(n)}(x)) \in \mathcal{S}_\Delta. \tag{1.3}$$

2. Prolongation of vector fields and infinitesimal symmetries

2.1. Prolongation of local diffeomorphisms.

Definition 2.1. Let M be an open subset of $X \times U$ and g a local diffeomorphism $M \to M$. Then $\mathsf{pr}^n g : M^{(n)} \to M^{(n)}$, the n-th prolongation of g on $M^{(n)} = \{(x, u^{(n)}) : (x, u) \in M\}$ is defined as follows. For all $(x_0, u_0^{(0)}) \in M^{(n)}$, take any function $u = f(x)$ such that $(x_0, f^{(n)}(x_0)) = (x_0, u_0^{(n)})$ and let $\tilde{u}(\tilde{x}) = (g \circ f)(\tilde{x})$. Then

$$\mathsf{pr}^n g(x_0, u_0^{(n)}) := (\tilde{x_0}, \tilde{u_0}^{(n)}(\tilde{x_0}))$$

, where $(\tilde{x_0}, \tilde{u_0}) = g(x_0, u_0)$. This is well-defined i.e. independent of choice of f.

Remark 2.2. In (x, u) space, 1-jet of $u = f(x)$ may be considered as slopes of some line elements in its graph. Transform image of this graph by a local diffromorphism g is put $\tilde{u} = \tilde{f}(\tilde{x})$ in new coordinates. Calculate the slopes of this new graph. The process of assigning new slopes to old slopes when the graph is being transformed by g is 1st prolongation of g in naive sense.

2.2. Prolongation of group actions.

Let G be a local group of transformation acting on M. Then $\mathsf{pr}^n G := \{\mathsf{pr}^n g : g \in G\}$ acts on $M^{(n)}$.

Example 2.3. Example1.2 continued. Suppose that $\mathsf{pr}^1\Theta : X \times U^{(1)} \to X \times U^{(1)}$ sends $(x_0, u_0, u_0') \to (\tilde{x}, \tilde{u}, \tilde{u}')$. Then $\mathsf{pr}^1\theta(x_0, u_0, u_0') = (x_0 \cos\theta - u_0 \sin\theta, x_0 \sin\theta + u \cos\theta, \frac{u_0' \cos\theta + \sin\theta}{\cos\theta - u_0' \sin\theta})$. Dropping 0 subscripts we have generally

$$\mathsf{pr}^1\theta(x, u, u') = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{u_x\cos\theta + \sin\theta}{\cos\theta - u_x\sin\theta})$$

2.3. Prolongation of Vector fields.

Definition 2.4. Let M be an open subset of $X \times U$. Let V be an vector field on M and $\varphi_\varepsilon := \exp(\varepsilon V)$ is 1 parameter group of local diffeomorphisms, which are *flows*. Then the prolongation of vector field V, $\mathsf{pr}^n V$ is a vector field on $M^{(n)}$ defined by

$$\mathsf{pr}^n V(x, u^{(n)}) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathsf{pr}^n(\exp\varepsilon V)(x, u^{(n)}).$$

Example 2.5. Let $p = q = 1$, $X = \mathbb{R}$ and $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$. Then $\exp(\varepsilon V)$ is a rotation by angle ε which is calculated as follows. Noting $V = (-u, x)$,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x(\varepsilon) \\ u(\varepsilon) \end{pmatrix} = e^{\varepsilon A} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$e^{\varepsilon A} = I + \varepsilon A + \frac{\varepsilon^2}{2}A^2 + \cdots = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}.$$

Its action on jets is given by

$$\begin{aligned} \mathsf{pr}^1 V(x,u,u_x) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathsf{pr}^1 \exp(\varepsilon V)(x,u,u_x) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (x\cos\varepsilon - u\sin\varepsilon, x\sin\varepsilon + u\cos\varepsilon, \frac{u_x\cos\varepsilon + \sin\varepsilon}{\cos\varepsilon - u_x\sin\varepsilon}) \\ &= (-u, x, 1+u_x^2). \end{aligned}$$

Hence $\mathsf{pr}^1 V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}$.

Example 2.6. Given $u(x,y)$ and Laplace equation $u_{xx} + u_{yy} = 0$. 2nd jet space is $\{(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})\} \subset X \times U^{(2)} \subset \mathbb{R}^8$. Let the equation be $\Delta(x,u^{(2)}) := u_{xx} + u_{yy} = 0$ then $\mathcal{S}_\Delta = \{\Delta = 0\}$ is a hypersurface since Δ is of maximal rank on its zero set with the Jacobian $(0,\ldots,0,\underbrace{1}_{6th},0,1)$.

2.4. Symmetry groups of partial differential equations.

Definition 2.7. Let G be a local group of transformations acting on $X \times U$ and $\Delta = 0$ with $\Delta = (\Delta_1,\ldots,\Delta_l)$ be a system of partial differential equations of order n. G is a symmetry group of $\Delta = 0$ if $\mathsf{pr}^n g$ sends $\mathcal{S}_\Delta$ into $\mathcal{S}_\Delta$ for every $g \in G$ or equivalently,

$$\mathsf{pr}^n V(\Delta_\nu) = 0 \text{ on } \mathcal{S}_\Delta$$

for every $\nu = 1,2,\ldots,l$ and every infinitesimal generator V of G.

Definition 2.8. By a differential function of order k we mean a $\mathcal{C}^\infty$ function $P(x,u^{(n)})$ defined on an open subset of $X \times U^{(n)}$. By the *total derivative* of P we mean

$$D_i P = D_{x_i} := \frac{\partial P}{\partial x_i} + \sum_{\substack{\alpha=1,\ldots,q \\ |J|\le n}} \frac{\partial P}{\partial u_J^\alpha} u_{J,i}^\alpha.$$

The total derivative of P is a differential function of order $n+1$.

Example 2.9. Let $u(x,y)$ be defined on $\mathbb{R}^2$ then a tota derivative

$$D_x(xu + u_x + u_y^2) = u + xU_x + u_{xx} + 2u_y u_{xy}$$